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FLEXURAL VIBRATIONS OF CIRCULAR BEAMS

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For circular cylindrical beams, the Timoshenko theory, which includes shear and rotary effects, can be applied by introducing a constant $K' = 0{\cdot}9$. The study presented here takes into account the actual configuration and thus makes it an integral part of the analysis. A variational approach previously used [7] is followed. A simplification of the three-dimensional problem is obtained using the inverse method due to St. Venant. A biquadratic equation yields numerical results for the natural frequencies of the first three modes for the following cases: simply supported, fixed–fixed, free–free and fixed–free.

1. INTRODUCTION

The flexural vibrations of long thin cylindrical rods are well described by the elementary Bernoulli–Euler (B–E) theory. The effect of rotary inertia is included in the theory of the so-called Rayleigh beam [1]. The Timoshenko beam model [10], which includes shear and rotary effects, has been the object of study since its appearance in the literature [2]. The simplifications introduced in the B–E, Rayleigh and Timoshenko theories reduced the problem from three dimensions to a single dimension. In order to apply the Timoshenko theory to various cross-sections, a shear correction factor K', which is an inherent part of the theory, has been introduced. For a circular section, the value K' is generally taken as $0{\cdot}9$.

The basis for the determination of the natural frequencies of a vibrating circular cylinder has been the exact solution of the three-dimensional equations of elasticity by Pochhammer and Chree [3]. Solutions for the free–free condition have been presented by Hutchinson [4] and Pickett [5]. The Pochhammer–Chree solutions satisfied the traction free boundary conditions on the curved surfaces but, when applied to a finite cylinder, carried with them a requirement for the existence of stresses and displacements on the plane terminal faces which seldom occur in practice. An exact solution satisfying all types of boundary conditions at the terminal has as yet not been effected. Hutchinson satisfied the boundary conditions on the terminals approximately by using a condition of orthogonality between stresses and displacements. Pickett's approximate solution specified that the total shear and moment on the terminal faces be zero. Recently, Leissa and So [6] have presented a three-dimensional solution based on a Rayleigh–Ritz procedure. Fourier series and polynomials were employed to describe the displacement functions. Two cases were covered: free–free and fixed–free boundary conditions. Solutions for the simply supported and fixed–fixed conditions of the one-dimensional theories were not given.

A variational approach [7] is used herein by applying Hamilton's principle of least action. After simplification of the three-dimensional problem in accordance with

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the inverse method due to St. Venant, the present study involves the determination of the transverse and longitudinal displacements of the vibrating body. The application of the principle yields two Euler differential equations defining admissible displacement functions along with associated natural boundary conditions.

2. SIMPLIFICATION OF THE THREE-DIMENSIONAL PROBLEM

The circular cylinder is shown with respect to the XYZ co-ordinate system in Figure 1. The X-axis is taken as the central line of the beam and Y- and Z-axes lie in the principal planes of the cross-sections at their centroid. The vibration is assumed to take place in the principal X–Y plane with the displacements u and v as functions of (x, y, t). The displacement w in the direction of the Z-axis leading to a distortion of the cross-sections is neglected. The body forces are to be taken as zero. The curved surfaces are unloaded.

Uncer certain conditions, simplifying assumptions may be made as to the relative importance of certain internal stresses during small vibrations about the central axis of equilibrium. These assumptions are inherent in the one-dimensional theories. The process of assigning zero values to certain stresses is closely connected to the inverse method used by St. Venant in his solutions for the bending of prismatic beams. Their actual existence can be of importance under certain conditions, and solutions based on their neglect may be of limited application. Thus it is to be expected that such assumptions may not be applied for deep beams or for high modes of vibration.

With flexure confined to that about the OZ axis, the normal stress σ_z and shear stress τ_{zy} are both taken as zero. In the plane normal to the X-axis, the lines of shearing stress have the component τ_{xy} and τ_{xz}. The τ_{xz} stress exists due to the condition of a curved surface. However, the resultant shear force in the OZ direction is zero. The magnitude of τ_{xz} is taken as zero. With these simplifications we have a stress system consisting of σ_x, σ_y and τ_{xy}. Thus the cylinder is assumed to behave as if it were composed of an infinite number of lamina of varying depth acting independently and vibrating in unison in a state of plane stress.

3. DISPLACEMENT FUNCTIONS

The displacement functions u and v are considered to be unknown functions of the spatial co-ordinates x and y along with the time t. The displacements call not only for flexural strains but also include simultaneous shearing strains. The displacement functions are taken as

$$u(x, y, t) = \alpha'(x, t)y - \tfrac{1}{3}\beta'(x, t)y^3; \; v(x, y, t) = -\alpha(x, t) + R^2\beta(x, t). \qquad (1, 2)$$

The symbol ($'$) represents differentiation with respect to the variable x.

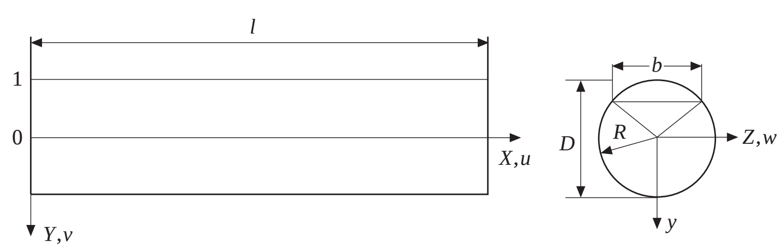

Figure 1. The co-ordinate system.

The u displacement is composed of a linear term in y and a term varying as y^3, thus introducing a correction to the usual assumption that planes remain plane as expressed by the term $\alpha' y$. The v displacement is independent of y and so is assumed to be true for all y. It follows from equations (1) and (2) that the strains are given by

$$\varepsilon_x = \partial u/\partial x = \alpha'' y - \tfrac{1}{3}\beta'' y^2, \qquad \varepsilon_y = \partial v/\partial y = 0, \tag{3a, b}$$

$$\gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = \beta'(R^2 - y^2). \tag{4}$$

From Hooke's law, the vertical component τ_{xy} of the shear stress varies parabolically over the depth of the section coinciding with the law of elementary strength of materials.

4. POTENTIAL ENERGY AND KINETIC ENERGY FUNCTIONALS

The potential strain energy per unit of volume consistent with the stress system σ_x, σ_y, τ_{xy} is taken as

$$V = \frac{E}{2(1 - v^2)}(\varepsilon_x^2 + \varepsilon_y^2 + 2v\varepsilon_x\,\varepsilon_y) + \frac{G}{2}\gamma_{xy}^2, \tag{5a}$$

in which v is the Poisson ratio. Neglecting the effect of σ_y, ε_y is taken as $\varepsilon_y = -v\varepsilon_x$:

$$V = \frac{E}{2}\varepsilon_x^2 + \frac{G}{2}\gamma_{xy}^2. \tag{5b}$$

The kinetic energy per unit of volume is

$$T = (\rho/2)\,[(\partial u/\partial t)^2 + (\partial v/\partial t)^2]. \tag{6}$$

Integrating over the circular cross-section, the internal strain energy and kinetic energy per unit of dimension are, respectively,

$$V = EI[\tfrac{1}{2}(\alpha'') - \tfrac{1}{6}\alpha''\beta'' R^2 + \tfrac{5}{288}(\beta'')^2 R^4] + \tfrac{5}{4}AR^4 G(\beta')^2, \tag{7}$$

$$T = \rho\tfrac{1}{2}\{(\alpha_{xt})^2 - \tfrac{1}{3}\alpha_{xt}\,\beta_{xt}\,R^2 + \tfrac{5}{144}(\beta_{xt})^2 R^4\} + \frac{A\rho}{2}\{\alpha_t^2 - 2\alpha_t\,\beta_t\,R^2 + \beta_t^2\,R^2\}, \tag{8}$$

where $I = \Pi R^4/4$ and $A = \Pi R^2$.

5. APPLICATION OF HAMILTON'S PRINCIPLE

A fundamental approach to the study of the flexure of the circular cylinder is the Hamilton principle of least action:

$$\delta J = \delta \int_{t_1}^{t_2} (T - V)\,\mathrm{d}t = 0,$$

which, when applied to the vibrating beam, becomes

$$\delta \int_{t_1}^{t_2} \mathrm{d}t \int_0^l \mathrm{d}x \int_{-R}^{+R} (T - V) b\,\mathrm{d}y = 0, \tag{9}$$

where $b = 2\sqrt{R^2 - y^2}$.

Using the values of T and V of equations (5) and (6), the statement from Hamilton's principle becomes

$$\int_{t_1}^{t_2} \mathrm{d}t \int_0^l \{L_1(\alpha) - R^2 L_2(\beta)\}\delta\alpha \,\mathrm{d}x + R^2 \int_{t_1}^{t_2} \mathrm{d}t \int_0^l \{L_3(\alpha) - R^2 L_4(\beta)\}\delta\beta \,\mathrm{d}x$$

$$+ \left[\left(EI\alpha^{IV} - \frac{EI}{6} R^2\beta'''\right)\delta\alpha\right]_0^l + \left[\left(-EI\alpha'' + \frac{E}{6} R^2 I\beta''\right)\delta\alpha'\right]_0^l$$

$$+ \left[\left(\tfrac{5}{144} EIR^4\beta''' - EI\frac{R^2}{6}\alpha''' - \tfrac{5}{8} GR^4 A\beta'\right)\delta\beta\right]_0^l$$

$$+ \left[\left(\frac{E}{6} R^2 I\alpha'' - \tfrac{5}{144} EIR^4\beta''\right)\delta\beta'\right]_0^l = 0, \tag{10}$$

where

$$L_1(\alpha) = \left[EI\frac{\partial^4}{\partial x^4} - \rho I\frac{\partial^4}{\partial x^2\,\partial t^2} + A\rho\frac{\partial^2}{\partial t^2}\right]\alpha, \tag{11a}$$

$$L_2(\beta) = \left[\frac{EI}{6}\frac{\partial^4}{\partial x^4} - \frac{\rho I}{6}\frac{\partial^4}{\partial x^2\,\partial t^2} + A\rho\frac{\partial^2}{\partial t^2}\right]\beta, \tag{11b}$$

$$L_3(\alpha) = \left[\frac{EI}{6}\frac{\partial^4}{\partial x^4} - \frac{\rho I}{6}\frac{\partial^4}{\partial x^2\,\partial t^2} + A\rho\frac{\partial^2}{\partial t^2}\right]\alpha, \tag{11c}$$

$$L_4(\alpha) = \left[\frac{5EI}{144}\frac{\partial^4}{\partial x^4} - \frac{5\rho I}{144}\frac{\partial^4}{\partial x^2\,\partial t^2} + A\rho\frac{\partial^2}{\partial t^2} - \tfrac{5}{8} AG\frac{\partial^2}{\partial x^2}\right]\beta. \tag{11d}$$

The Euler equations are

$$L_1(\alpha) - R^2 L_2(\beta) = 0, \qquad L_3(\alpha) - R^2 L_4(\beta) = 0. \tag{12, 13}$$

The terms at the limits of equation (10) furnish both forced and natural boundary conditions.

6. BOUNDARY CONDITIONS

Boundary conditions play a prominent part in the determination of the natural frequencies of vibrating beams [7]. Forced boundary conditions such as zero deflection and zero slope at a support located at $x = 0$, for example, require that for all values of y,

$$\alpha(0) = 0, \qquad \beta(0) = 0, \qquad \alpha'(0) = 0, \qquad \beta'(0) = 0. \tag{14a–d}$$

Accordingly, $\delta\alpha = \delta\alpha' = \delta\beta = \delta\beta' = 0$ and the terms at the limits of equation (10) vanish.

Natural boundary conditions result from the vanishing of the terms at the limits by equating the coefficients of $\delta\alpha$, $\delta\alpha'$, $\delta\beta$ and $\delta\beta'$ to zero. The hinged boundary condition is

$$EI\alpha''(0) - \frac{E}{6} R^2 I\beta''(0) = 0, \qquad \frac{E}{6} R^2 I\alpha''(0) - \tfrac{5}{144} EIR^4\beta''(0) = 0, \tag{15a, b}$$

which leads to

$$\alpha''(0) = \beta''(0) = 0. \tag{16}$$

For a free–free condition,

$$EI\alpha'''(0) - \frac{EI}{6} R^2\beta'''(0) = 0, \tag{17a}$$

$$\tfrac{5}{144} EIR^4\beta'''(0) - \frac{EIR^2}{6} \alpha'''(0) - \tfrac{5}{8} GR^4 A\beta'(0) = 0, \tag{17b}$$

$$EI\alpha''(0) - \frac{ER^2}{6} I\beta''(0) = 0, \tag{17c}$$

$$\frac{ER^2}{6} I\alpha''(0) - \tfrac{5}{144} EIR^4\beta''(0) = 0, \tag{17d}$$

which leads to

$$\alpha''(0) = 0, \qquad \beta''(0) = 0, \qquad \alpha'''(0) = \frac{R^2}{6} \beta'''(0), \tag{18a}$$

$$EI\beta'''(0) - 90AG\beta'(0) = 0. \tag{18b}$$

Since a free–free condition requires $\sigma_x = 0$ and $\tau_{xy} = 0$, for all values of y it follows from equations (3) and (4) that equations (18a,b) reduce to the conditions

$$\alpha''(0) = \alpha'''(0) = \beta'(0) = \beta''(0) = \beta'''(0) = 0. \tag{19a–e}$$

7. THE EULER EQUATIONS

When the terms at the limits of equation (10) vanish as a result of the forced and natural boundary conditions, all that remains is the integral term. The Euler equations are thus obtained for arbitrary variations of $\delta\alpha$ and $\delta\beta$. These homogeneous linear equations, together with the homogeneous boundary conditions (14)–(19), determine the eigenvalues and eigenfunctions of a boundary value problem. However, except for the simply supported case, formal solutions of the two Euler equations (12) and (13) with the associate boundary conditions are difficult, if not impossible, to obtain. Recourse must be taken to approximate methods of solution, such as the Galerkin [8] procedure, to obtain the practically important natural frequencies.

8. THE SIMPLY SUPPORTED BEAM

The boundary conditions for a simply supported beam at $x = 0$ and $x = l$ are

$$\alpha = 0, \qquad \beta = 0, \qquad \alpha'' = 0, \qquad \beta'' = 0.$$

Assuming harmonic vibrations, a particular solution of equations (12) and (13) that satisfies these boundary conditions is taken as

$$\alpha = C_m \sin Kx \cos p_m t, \qquad \beta = D_m \sin Kx \cos p_m t,$$

where $K = m\Pi/l$, m is an integer that expresses the mode number and p_m is the vibration frequency. The $\sin Kx$ are the eigenfunctions of the boundary value problem, provided that the following characteristic determinant is zero:

$$\Delta = \begin{vmatrix} [EIK^4 - p^2(\rho IK^2 + A\rho)] & \left[p^2\left(A\rho + \rho\dfrac{I}{6}K^2\right) - \dfrac{EI}{6}K^4\right] \\ \left[E\dfrac{IK^4}{6} - p^2\left(\rho\dfrac{IK^2}{6} + A\rho\right)\right] & [p^2(\frac{5}{144}\rho IK^2 + A\rho) - (\frac{5}{144}EIK^4 + \frac{5}{8}AGK^2)] \end{vmatrix} = 0.$$

This yields the characteristic bi-quadratic equation

$$ap^4 + bp^2 + c = 0. \tag{20}$$

The parameters to be used for finding the roots of equation (20) are

$$\frac{b}{a} = -\frac{(V_1/l^2)(m\Pi)^2}{101 + K^2r^2}\left\{101 + 2K^2r^2 + \frac{90}{101 + K^2r^2}\frac{G}{E}(1 + K^2r^2)\right\},$$

$$\frac{c}{a} = (V_1/l)^4\frac{(m\Pi)^4}{101 + K^2r^2}\left\{90\frac{G}{E} + K^2r^2\right\},$$

where $V_1 = \sqrt{E/\rho}$, $r = \sqrt{I/A} = R/2$ and $Kl = m\Pi$.

Numerical results for $G/E = 3/8$ are given in Table A.2 of Appendix A.

9. GALERKIN TYPE SOLUTIONS

The Galerkin procedure consists of choosing a class of admissible co-ordinate functions so that the forced and/or natural boundary conditions are satisfied or that the terms at the limits vanish. In general, these functions will not be the solutions of the differential equations that define the eigenvalue problem. The weighted error obtained by substituting these pseudo-eigenfunctions into the left side of the Euler equations is integrated over the range (0, l) and equated to zero. The weighting functions are the pseudo-eigenfunctions. A system of equations containing undetermined coefficients associated with the co-ordinate functions is obtained. The determinant of this system is equated to zero and yields the characteristic equation for the eigenvalues.

It is possible to choose the co-ordinate functions on the basis of knowledge of the eigenfunctions of a system with slightly different characteristics, but which satisfy the same boundary conditions. The choice of co-ordinate functions is considerably eased in the cases of beams that are supported at the terminals, due to the fact that the $\beta(x)$ function must satisfy the same boundary conditions as the $\alpha(x)$ function. The $\alpha(x)$ functions pertain to the one-dimensional B–E model and the eigenvalues for this case are known [9].

In contrast to the Rayleigh–Ritz method, in which the vibration mode is taken in the form of a series of admissible functions, only one term with an undetermined coefficient is taken here, thus leading to a considerable reduction in computation. This single term is the B–E solution for the desired mode, and a meaningful frequency is obtained from the solution of the bi-quadratic equation.

10. THE END SUPPORTED BEAM

In addition to the simply supported (SS) beam treated in section 8, classical cases treated in engineering literature include the fixed–fixed (FF) and fixed supported (FS) beams. Co-ordinate functions for the (FF) and (FS) cases are chosen to be

$$\alpha = C_m f_m(x)\cos p_m t, \qquad \beta = D_m f_m(x)\cos p_m t,$$

where the $f_m(x)$ are eigenfunctions of the B–E model and m represents the mode numbers of the vibration mode. Since the terms at the limits vanish, we obtain, from equation (10),

$$\int\int\{L_1(\alpha)-R^2L_2(\beta)\}\delta C_m f_m(x)\cos p_m t\,\mathrm{d}x\,\mathrm{d}t=0, \tag{21}$$

$$\int\int\{L_3(\alpha)-R^2L_4(\beta)\}\delta D_m f_m(x)\cos p_m t\,\mathrm{d}x\,\mathrm{d}t=0. \tag{22}$$

A condition for a solution other than $C_m=D_m=0$ is provided by equating the characteristic determinant to zero:

$$\begin{vmatrix} a_{11} & a_{12}\\ a_{21} & a_{22}\end{vmatrix}=a_{11}\,a_{22}-a_{12}\,a_{21}=0,$$

where

$$a_{11}=\int_0^l[EIf_mf_m^{IV}+\rho Ip_m^2f_mf_m''-A\rho p_m^2f_m^2]\,\mathrm{d}x,$$

$$a_{12}=R^2\left(\int_0^l\left[A\rho p_m^2f_m^2-\frac{\rho}{6}Ip_m^2f_mf_m''-\frac{\rho}{6}If_mf_m^{IV}\right]\mathrm{d}x\right),$$

$$a_{21}=\int_0^l[EIf_mf_m^{IV}+\rho Ip_m^2f_mf_m''-6A\rho p_m^2f_m^2]\,\mathrm{d}x,$$

$$a_{22}=R^2\left(\int_0^l[-\tfrac{5}{144}EIf_mf_m^{IV}-(\tfrac{5}{144}\rho Ip_m^2-\tfrac{5}{8}AG)f_mf_m''+A\rho p_m^2f_m^2]\,\mathrm{d}x\right).$$

The value of p_m is obtained from the bi-quadratic equation

$$a_{11}\,a_{22}-a_{12}\,a_{21}=ap_m^4+bp_m^2+c=0.$$

11. THE FIXED–FIXED (FF) BEAM

The B–E eigenfunction for the FF case is [9]

$$f_m(x)=(\mathrm{ch}\,\beta_m x-\cos\beta_m x)-\alpha_m(\mathrm{sh}\,\beta_m x-\sin\beta_m x),$$

where $\beta_m l$ are eigenvalues and where

$$\alpha_m=(-\cos\beta_m l+\mathrm{ch}\,\beta_m l)/(-\sin\beta_m l+\mathrm{sh}\,\beta_m l).$$

The orthogonality of the eigenfunctions leads to

$$\int_0^l f_m^2=l,\qquad \int_0^l f_mf_m''\,\mathrm{d}x=\alpha_m\,\beta_m\,(2-\alpha_m\,\beta_m\,l)=\phi_m,$$

$$\int_0^l f_mf_m^{IV}\,\mathrm{d}x=\beta_m^4\int_0^l f_m^2\,\mathrm{d}x=\beta_m^4\,l.$$

Felgar [9] furnishes numerical values of β_m together with formulas for the ϕ_m functions. It follows that

$$a_{11} = p_m^2\,[\rho I\phi_m - A\rho l] + EI\beta_m^4\,l,$$

$$a_{12} = \frac{R^2}{6}\,[p_m^2\,(6A\rho l - \rho I\phi_m)] - EI\beta_m^4\,l,$$

$$a_{21} = p_m^2\,[\rho I\phi_m - 6A\rho l] + EI\beta_m^4\,l,$$

$$a_{22} = R^2[-\tfrac{5}{144}\,EI\beta_m^4\,l + p_m^2\,(A\rho l - \tfrac{5}{144}\,\rho I\phi_m + \tfrac{5}{8}\,AG\phi_m)].$$

A meaningful value of p_m is obtained from

$$a_{11}\,a_{22} - a_{12}\,a_{21} = ap_m^4 + bp_m^2 + c = 0,$$

$$p_m = \left[\frac{-b/a \pm \sqrt{(b/a)^2 - 4(c/a)}}{2}\right]^{1/2},$$

$$b/a = (V_1\,/l)^2(N_1\,/D), \qquad c/a = (V_1\,/l)^4(r/l)^2(\beta_m\,l)^4(N_2\,/D), \qquad V_1 = \sqrt{E/\rho},$$

$$N_1 = \left\{\tfrac{5}{8}\,(r^2\phi_m^2 - \phi_m\,l)\,\frac{G}{E} - \tfrac{139}{144}\left(\frac{r}{l}\right)^2(\beta_m\,l)^4 + \tfrac{38}{144}\left(\frac{r}{l}\right)^4\phi_m\,l(\beta_m\,l)^4\right\}$$

$$N_2 = \tfrac{5}{8}\,\phi_m\,l\,\frac{G}{E} + \tfrac{19}{144}\left(\frac{r}{l}\right)^2(\beta_m\,l)^4,$$

$$D = 5 - \tfrac{139}{144}\left(\frac{r}{l}\right)^2(\phi_m\,l) - \tfrac{38}{144}\left(\frac{r}{l}\right)^4(\phi_m\,l)^2.$$

For the first three modes the values of $\phi_m\,l$ and $(\beta_m\,l)^4$ are shown in Table A.1 and values for the eigenvalues in Table A.3, both in Appendix A.

12. THE FIXED–FREE AND FREE–FREE BEAMS

The fixed–free condition is found in the case of the cantilevered beam. Pochhammer's [3] three-dimensional theory is used as the basis for the study of the free–free condition.

At a free end, $x = l$, the boundary conditions require that the shear and normal tractions must vanish. The one-dimensional solutions require that the total moment and shear be zero. If $v(x)$ denotes the total transverse deflections, the B–E conditions are $v''(l) = v'''(l) = 0$. Timoshenko's theory requires that the curvature of the central axis at $x = l$ due to pure bending be zero and that the total shear also be zero. The Timoshenko boundary conditions involve two independent functions of x. Pickett's [5] boundary conditions are expressed by two integrals, as do those of Hutchinson [4]. The boundary conditions of the present study are given by equations (17) and, equivalently to the traction free conditions, $\sigma_x\,(l) = \tau_{xy}\,(l) = 0$. Thus

$$\alpha''(l) = \alpha'''(l) = \beta'(l) = \beta''(l) = \beta'''(l) = 0. \tag{23}$$

For the free–free case, these conditions must also be satisfied at $x = 0$.

The pseudo-eigenfunctions $f_{\alpha m}\,(x)$ for the dominant $\alpha(x)$ which represents the pure bending configuration are taken as identical with that of the B–E solution with identical

$v''(l) = v'''(l) = 0$. Timoshenko's theory requires that the curvature of the central axis at $x = l$ due to pure bending be zero and that the total shear also be zero. The Timoshenko boundary conditions involve two independent functions of x. Pickett's [5] boundary conditions are expressed by two integrals, as do those of Hutchinson [4]. The boundary conditions of the present study are given by equations (17) and, equivalently to the traction free conditions, $\sigma_x(l) = \tau_{xy}(l) = 0$. Thus

$$\alpha''(l) = \alpha'''(l) = \beta'(l) = \beta''(l) = \beta'''(l) = 0. \tag{23}$$

For the free–free case, these conditions must also be satisfied at $x = 0$.

The pseudo-eigenfunctions $f_{\alpha m}(x)$ for the dominant $\alpha(x)$ which represents the pure bending configuration are taken as identical with that of the B–E solution with identical peaks and nodes. They satisfy the boundary conditions at a supported terminal $x = 0$. At a free end, $x = l$, they yield the following values: $f''_{\alpha m}(l) = f'''_{\alpha m}(l) = 0$.

The pseudo-eigenfunctions for $\beta(x)$ denoted by $f_{\beta m}(x)$ must satisfy the forced boundary conditions at a supported terminal, $x = 0$. They must also furnish derivative values at $x = l$ so that the limit terms vanish.

Accordingly, we require that

$$f'_{\beta m}(l) = f''_{\beta m}(l) = f'''_{\beta m}(l) = 0.$$

To act as a correction to the dominant $\alpha(x)$ function they must additionally have the same number and location of nodes.

12.1. THE FREE–FREE BEAM

The pseudo-eigenfunction $f_{\alpha m}(x)$ for the free–free beam are taken [9] as

$$f_{\alpha m}(x) = \mathrm{ch}\, \beta_m x + \cos \beta_m x - \alpha_m (\mathrm{sh}\, \beta_m x + \sin \beta_m x),$$

where $\beta_m l$ are eigenvalues and where $\alpha_m = (\mathrm{ch}\, \beta_m l - \cos \beta_m l)/(\mathrm{sh}\, \beta_m l - \sin \beta_m l)$. At the free ends they satisfy the conditions $f''_{\alpha m}(0) = f'''_{\alpha m}(0) = f''_{\alpha m}(l) = f'''_{\alpha m}(l) = (0)$. Proposed pseudo-eigenfunctions for $\beta(x)$ are trigonometrical polynomials. The first and third modes are symmetrical with respect to the centreline. The second mode is anti-symmetrical. From equation (23) they must satisfy the conditions $f'_{\beta m}(0) = f''_{\beta m}(0) = f'''_{\beta m}(0) = f'_{\beta m}(l) = f''_{\beta m}(l) = f'''_{\beta m}(l) = (0)$.

For the first and third modes $f_{\beta m}(x)$ are, respectively, taken as

$$f_{\beta 1}(x) = 1 + 12{\cdot}8733 \cos \frac{2\Pi x}{l} + 3{\cdot}218325 \cos \frac{4\Pi x}{l},$$

$$f_{\beta 3}(x) = 1 - 26{\cdot}35976 \cos \frac{2\Pi x}{l} + 30{\cdot}831439 \cos \frac{4\Pi x}{l} - 10{\cdot}773999 \cos \frac{6\Pi x}{l}.$$

For the second mode,

$$f_{\beta 2}(x) = 0{\cdot}3195885\left(1 - 2\frac{x}{l}\right) - 4{\cdot}864362 \sin \frac{2\Pi}{l} x + 3{\cdot}9830453 \sin \frac{4\Pi x}{l} + \sin \frac{6\Pi x}{l}.$$

The pseudo-functions also furnish nodes that coincide with those of the B–E solutions.

12.2. THE FIXED–FREE BEAM

From Felgar [9] we obtain the pseudo-eigenfunctions for the cantilever beam,

$$f_{\alpha m}(x) = (\mathrm{ch}\, \beta_m x - \cos \beta_m x) - \alpha_m (\mathrm{sh}\, \beta_m x - \sin \beta_m x),$$

together with the numerical values of the eigenvalues. Zero slope and deflection at $x = 0$ are satisfied. Zero traction values at $x = l$ are satisfied by $f''_{\alpha m}(l) = f'''_{\alpha m}(l) = 0$. Pseudo-eigenfunctions for $\beta(x)$ are as follows for the first three modes:

$$f_{\beta 1}(x) = -1{\cdot}25 + \cos\frac{\Pi x}{l} + 0{\cdot}25\cos\frac{2\Pi x}{l},$$

$$f_{\beta 2}(x) = 1{\cdot}5056291 + \cos\frac{\Pi x}{l} - 1{\cdot}6577433\cos\frac{2\Pi x}{l} - 0{\cdot}8478859\cos\frac{3\Pi x}{l}.$$

$$f_{\beta 3}(x) = -1{\cdot}2356298 + \cos\frac{\Pi x}{l} - 0{\cdot}728263\cos\frac{2\Pi x}{l} + 0{\cdot}4603693\cos\frac{3\Pi x}{l}$$
$$+ 0{\cdot}503352\cos\frac{4\Pi x}{l},$$

which satisfy $f'_{\beta m}(l) = f''_{\beta m}(l) = f'''_{\beta m}(l) = 0$ and furnish the required number and location of nodes.

The same procedure for determining the natural frequencies is identical with that outlined in section 10 for the end supported beams, with the values a_{11}, a_{12}, a_{21} and a_{22} determined from

$$a_{11} = \int_0^l [EIf_{\alpha m} f^{IV}_{\alpha m} + \rho I p^2_m f_{\alpha m} f''_{\alpha m} - A\rho p^2_m f^2_m]\,\mathrm{d}x,$$

$$a_{12} = R^2\left(\int_0^l \left[A\rho p^2_m f_{\alpha m} f_{\beta m} - \frac{\rho}{6} I p^2_m f_{\alpha m} f''_{\beta m} - \frac{\rho}{6} I f_{\alpha m} f^{IV}_{\beta m}\right]\mathrm{d}x\right),$$

$$a_{21} = \int_0^l [EIf^{IV}_{\alpha m} f_{\beta m} + \rho I p^2_m f''_{\alpha m} f_{\beta m} - 6A\rho p^2_m f_{\alpha m} f_{\beta m}]\,\mathrm{d}x,$$

$$a_{22} = R^2\left(\int_0^l [-\tfrac{5}{144} EIf^{IV}_{\beta m} f_{\beta mm} - (\tfrac{5}{144}\rho I p^2_m - \tfrac{5}{8} AG) f_{\beta m} f''_{\beta m} + A\rho p^2_m f^2_{\beta m}]\,\mathrm{d}x\right).$$

Numerical values for the frequencies are given in Tables A.4 and A.5 of Appendix A.

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APPENDIX A: TABLES

Table A.1

Eigenvalues for a fixed–fixed B–E model beam

	$m=1$	$m=2$	$m=3$
$\phi_m l$	12·302617	46·050117	98·838405
$(\beta_m^l)^4$	500·56388	3803·5369	14617·624

Table A.2

Values of pl/V and p/p_0 for a simply supported beam; $p_0 = (m\Pi)^2(r/l)(V/l)$

mr/l	l/Dm	$pl/m\Pi V$	p/p_0	pl/Vm
0·00625	40	0·01959	0·9978	0·06154
0·0125	20	0·03918	0·9978	0·12309
0·025	10	0·07760	0·9880	0·23479
0·05	5	0·15004	0·9552	0·47136
0·075	3·33	0·21400	0·9083	0·67230
0·1	2·5	0·26847	0·8546	0·84342

Table A.3

Values for pl/V and p/p_0 for a fixed–fixed beam; $p_0 = (Kl)^2(r/l)(V/l)$, *where Kl are the classical values* [10] (4·370, 7·853, 10·996)

		First mode		Second mode		Third mode	
r/l	l/D	pl/V	p/p_0	pl/V	p/p_0	pl/V	p/p_0
0·00625	40	0·1379	0·9863	0·3749	0·9777	0·7212	0·9542
0·0125	20	0·2663	0·9523	0·7055	0·9152	1·3184	0·8723
0·025	10	0·4849	0·8671	1·2318	0·7990	2·2200	0·7345
0·05	5	0·8163	0·7298	1·9931	0·6464	3·4325	0·5678
0·075	3·33	1·0728	0·6394	2·5732	0·5563	4·2650	0·4764
0·1	2·5	1·2870	0·5731	3·0365	0·5202	4·6203	0·3821

TABLE A.4

Values for pl/V and p/p_0 for a free–free beam; $p_0 = (Kl)^2(r/l)(V/l)$, *where Kl are the classical values* [10] (4·370, 7·853, 10·996)

		First mode		Second mode		Third mode	
r/l	l/D	pl/V	p/p_0	pl/V	p/p_0	pl/V	p/p_0
0·00625	40	0·13966	0·9988	0·3828	0·9931	0·73511	0·9728
0·0125	20	0·27853	0·9960	0·75063	0·9737	1·3902	0·9198
0·025	10	0·5536	0·9843	0·4089	0·9138	2·4557	0·8124
0·05	5	1·06878	0·9554	2·4134	0·7827	4·0624	0·6720
0·075	3·33	1·53989	0·9177	3·1369	0·6782	5·3441	0·5893
0·1	2·5	1·98276	0·8862	3·6962	0·6092	6·4312	0·5319

TABLE A.5

Values for pl/V and p/p_0 for a fixed–free beam; $p_0 = (Kl)^2(r/l)(V/l)$, *where Kl are the classical values* (1·875, 4·694, 7·855)

		First mode		Second mode		Third mode	
r/l	l/D	pl/V	p/p_0	pl/V	p/p_0	pl/V	p/p_0
0·00625	40	0·02197	0·9999	0·1356	0·9848	0·3778	0·9785
0·0125	20	0·04366	0·9934	0·2632	0·9557	0·71426	0·9261
0·025	10	0·0881	0·9774	0·4835	0·8797	1·2476	0·8088
0·050	5	0·1763	0·9471	0·8162	0·7409	1·9719	0·6392
0·075	3·33	0·2570	0·8967	1·0769	0·6517	2·4875	0·5376
0·1	2·5	0·3382	0·8700	1·3109	0·5949	2·9043	0·4707